Riesz's Lemma

Let X be a normed linear space and x be an element of X. Let M be a proper closed subspace in X, if there exists $\theta < 1$ ($\delta > 0$) then:

$\exists x \in X : \|x\| = 1$ and $d(x, M) \geq \theta$ ($\geq 1 - \delta$)

In other words, for every proper closed subspace M, one can always find a vector x on the unit sphere of X such that d(x, M) is less than and arbitrarily close to 1.

Motivation

In a finite dimensional Euclidean space there is for every proper subspace M a unit vector perpendicular to x. The distance of any point m of M to x is then at least one, the value one is reached for m = 0.

In a normed space the concept of "perpendicular standing" is not in general definable. With respect to this, the formulation of the lemma of Riesz is a meaningful generalization. Is it not obvious that outside of a subspace there are still vectors with a positive distance from it.

Proof

The proof is trivial for $\theta < 0 (\delta > 1)$

Contrapositive:

Let X be a normed linear space and M be a any subspace in X,

If $\exists \theta < 1 : \forall x \in X$ with $\|x\| = 1$ and $d(x, M) < \theta$ then: M is dense in X.