Projection Methods and Discretization $-$Summary.

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We want to solve $Ax=y$ with $A\in L(X,Y)$ and $y\in R(A)$.
A projection method is just a discratization method that converts $Ax=y$ into $A_l x_l=Q_l y$.
where $A_l=Q_l AP_l$, $P_l: X\rightarrow X_l$, $Q_l: Y\rightarrow Y_l$.
In order to solve $Ax=y$ we approximate $x^+=A^+ y$ by $x_l^+=A_l^+ Q_l y$
If $\{x_l^+\}\rightarrow x^+$ $\forall\, y\in R(A)$ then, the projection method is convergent.

For a method to be convergent, two properties must be satisfied:

1. $\{A_l^+ Q_l A\}_l$ must be uniformly bounded, i.e., $\|A_l^+ Q_l A\|\leqslant C_A \forall\, l\in\mathbb{N}$
2. The projections P_l must converge pointwise to 0 in $N(A)^{\perp}$
   i.e., $\lim_{l\rightarrow\infty} P_{N(A_l)} x=0$ $\forall\, x\in N(A)^{\perp}$

To satisfy condition 2, the spaces X_l and Y_l must be choosen sensibly.
For an exact $y\in R(A)$ there is convergence.

Taking a step further, now we consider $R_l=A_l^+ Q_l$ as regularization operator,
and we have the following estimate of the total error:

$$\|x_l^{\varepsilon}-x^+\|_X\leqslant\|R_l\|\varepsilon+\|x_l^+-x^+\|$$

where the last term converges to zero for convergent projection methods.
Concerning the first term of the righthand side of the inequality, we have:

$$\|R_l\|=\sup\{\tfrac{\|R_l y\|_X}{\|y\|_Y}, 0\neq y\in Y\}\geqslant\sup\{\tfrac{\|x_l\|_X}{\|A_l x_l\|_Y}, 0\neq x_l\in N(A_l)^{\perp}\}=\alpha_l$$

For ill-posed problems $\alpha_l\rightarrow\infty$ as $l\rightarrow\infty$, but if there exists a constant C_R such that
$\alpha_l\leqslant\|R_l\|\leqslant C_R\alpha_l$ for $l\rightarrow\infty$, then, the porjection method is **robust.**

If a projection method is convergent and robust we can choose $l^*=l^*(\varepsilon)$
such that $(\{R_l\}_l,\, l^*)$ is a regularization of A^+.

If we have this last regularization, the approximation of the solution is greatly close.

Alternatively, we can apply the projection method with a large l, and on top of that, apply a regularization method such as Tikhonov-Phillips, which will return us a decent solution too.