The Secretary Problem

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Outline

1. Classical Secretary Problem
   - Introduction
   - Case of three candidates
   - Generalization to n >3
2. Variants and other similar problems:
   - Unknown number of candidates.
   - Cardinal payoff variant
   - Choosing more than one candidate

Statement of the classical problem

- There is only one vacancy available.
- The candidates can be ordered from worst to best with no ties but they arrive in random order.
- We can only determine the relative ranks of the candidates as they arrive (We cannot tell their absolute values).
- The goal is to choose the very best candidate, no one else will do.
- Once an candidate is rejected, she is gone forever and cannot be recalled.
- The total number of candidates n is known beforehand.

Let's look into the particular case $n = 3$

(B, C, A) → (100, 50, 25)

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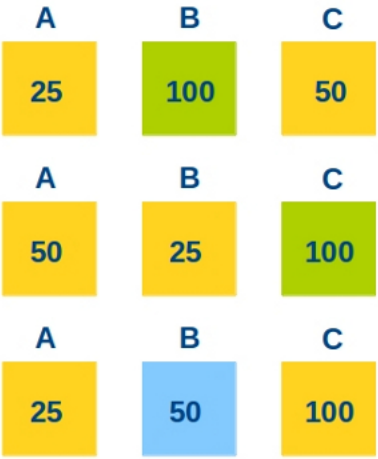
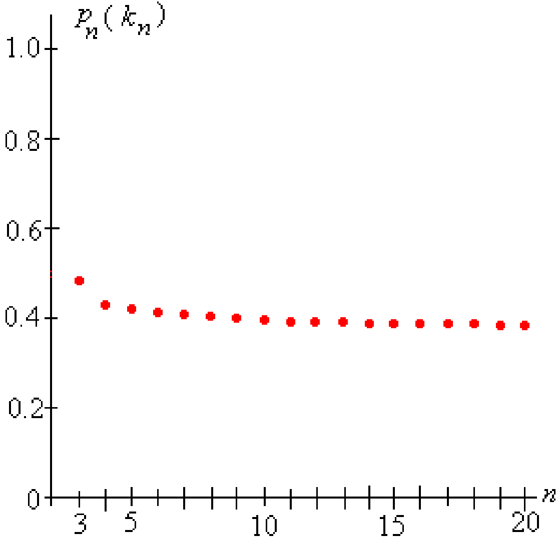
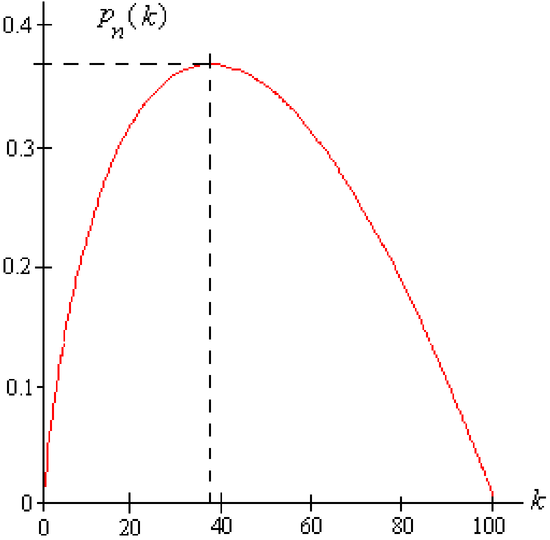


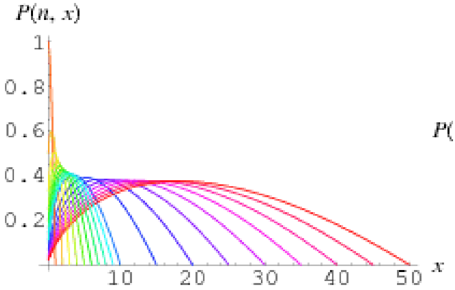
Número de cajas	3	4	5	6	7	8	9
Cajas desestimadas inicialmente	1	1	2	2	3	3	3
Éxito con estrategia	50%	45,8%	43,3%	42,8%	41,4%	41%	40,6%
Éxito sin estrategia	33%	25%	20%	16,5%	14,3%	12,5%	11,1%

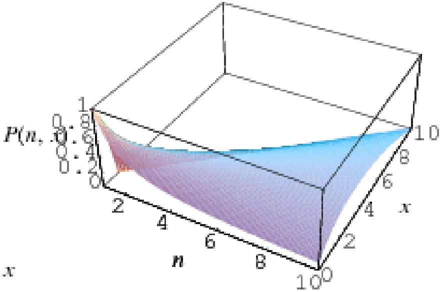
Probability of success versus n (using the optimal threshold k_n):



Given $n = 100$, probability of success depending on the threshold k:



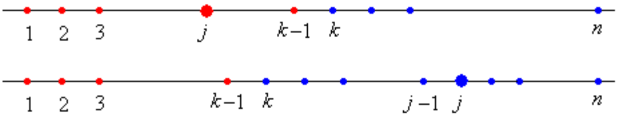




[1] x is the number of candidates, n the threshold.

For an arbitrary threshold k, the probability of choosing the best candidate is:

$$
\begin{aligned}
P(k) &= \sum_{j=1}^{n} P\left(\text{candidate } j \text{ is the best} \cap \text{candidate } j \text{ is selected}\right) \\
&= \sum_{j=1}^{n} P\left(\text{candidate } j \text{ is selected} | \text{candidate } j \text{ is the best}\right) \times P\left(\text{candidate } j \text{ is the best}\right) \\
&= \left[\sum_{j=1}^{k-1} 0 \times \frac{1}{n}\right] + \\
&+ \left[\sum_{j=k}^{n} P\left(\begin{array}{l} \text{the best of the firsts } j-1 \text{ candidates} \\ \text{is in the firsts } k-1 \text{ candidates} \end{array} \middle| \text{candidate } j \text{ is the best}\right) \times \frac{1}{n}\right] \\
&= \sum_{j=k}^{n} \frac{k-1}{j-1} \times \frac{1}{n} \quad = \quad \frac{k-1}{n} \sum_{j=k}^{n} \frac{1}{j-1}.
\end{aligned}
$$

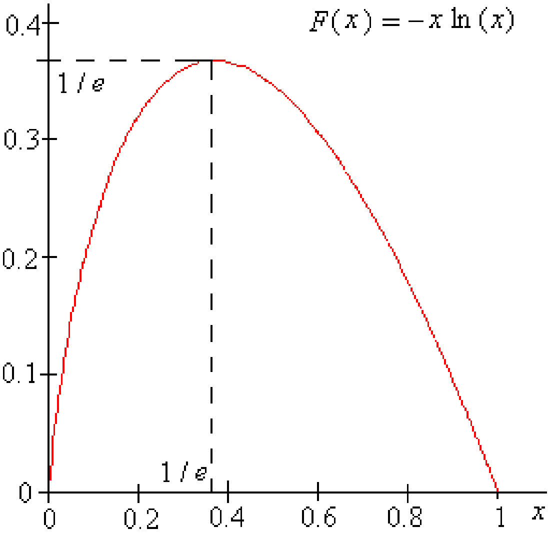


$$P(k) = \quad \frac{k-1}{n} \sum_{j=k}^{n} \frac{1}{j-1}$$

Simply letting n tend to infinity, writing x as the limit of k/n, denoting j/n as t and $1/n$ as dt, the sum above can be approximated by the integral:

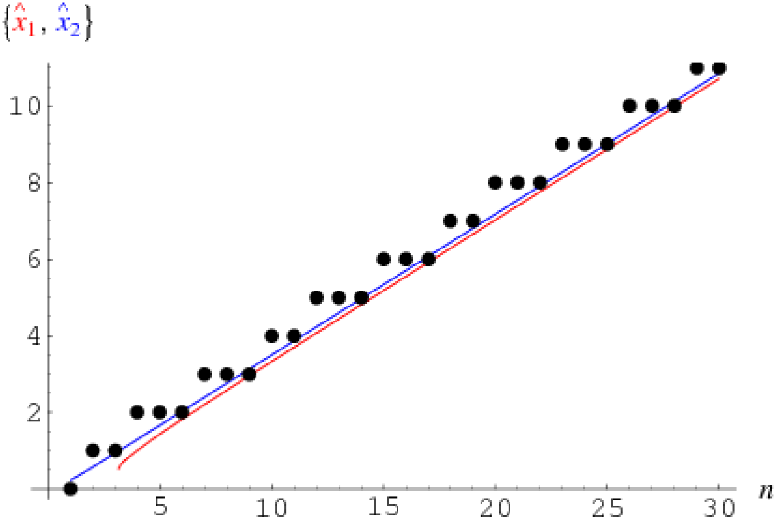
$$P(x) = x \int_x^1 \frac{1}{t}\, dt = -x \log(x).$$

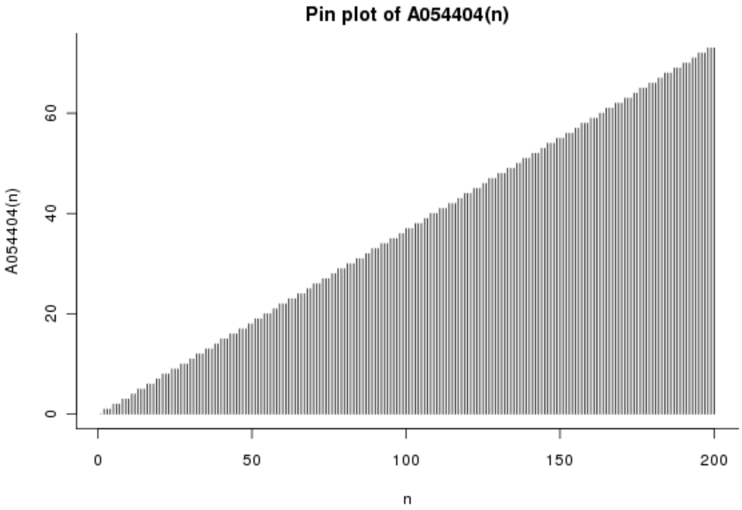
Deriving $P(x)$ with respect to x, positioning it at 0, and solving for x, we find that the optimal x is equal to e^{-1}. Thus, the optimal threshold tends to n/e as n increases, and the best candidate is selected with a probability of e^{-1}.

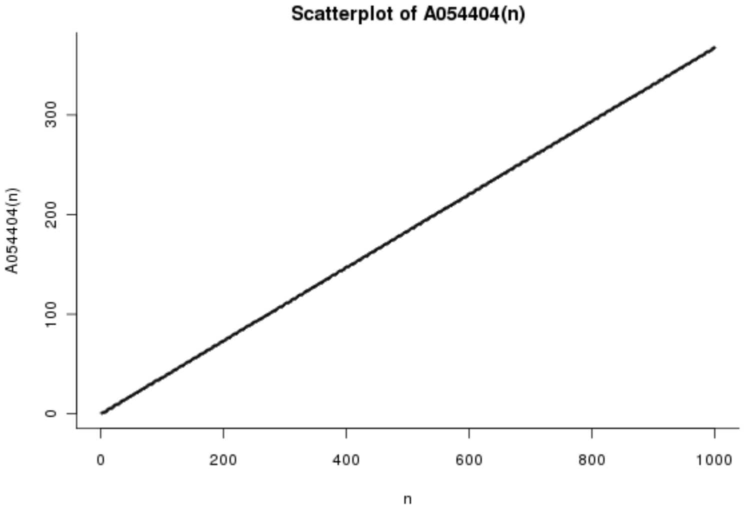

$F(x) = -x\ln(x)$
0.4
0.3
0.2
0.1
0
1 / e
1 / e
0
0.2
0.4
0.6
0.8
1
x

Número de cajas	3	4	5	6	7	8	9
Cajas desestimadas inicialmente	1	1	2	2	3	3	3
N/e	1,1036	1,4715	1,8394	2,2073	2,5752	2,943	3,3109
Éxito con estrategia	50%	45,8%	43,3%	42,8%	41,4%	41%	40,6%
Éxito sin estrategia	33%	25%	20%	16,5%	14,3%	12,5%	11,1%

The black dots represent (n, k_n):




Pin plot of A054404(n)
A054404(n)
0
20
40
60
0
50
100
150
200
n


Scatterplot of A054404(n)
A054404(n)
0
100
200
300
0
200
400
600
800
1000
n

Number of candidates n not known.

Unknown number of candidates.

If the value of n is not known beforehand. One way to remedy this is to assume that the number of candidates N is a random variable with a known distribution $P(N = k)_{k=1,2,\dots}$ (Presman and Sonin, 1972).

The essence of the model is based on that the problems in the real world appear in real time and it is easier to estimate times when events happen (arrival of candidates) than estimate the distribution of the specific number of events that will happen in total.

The model

A candidate must be selected in a time interval $[0, T]$ from an unknown number of valued candidates. The aim is to maximize the probability of selecting the best. Let's assume that all the candidates arrive independently with the same arrival time density f in $[0, T]$, and let F be the corresponding distribution function of the arrival time:

$$F(t) = \int_0^t f(s)ds, \; 0 \leq t \leq T.$$

The 1/e-law (1984)

Let τ be such that $F(\tau) = e^{-1}$. Consider the strategy to wait and observe all applicants up to time τ and then to select, if possible, the first candidate after time τ which is better than all preceding ones. Then this strategy, called 1/e-strategy, has the following properties:

1. for all N, there is a success probability of at least e^{-1}
2. is the unique strategy guaranteeing this lower success probability bound e^{-1}, and the bound is optimal.
3. selects, if there is at least one applicant, none at all with probability exactly e^{-1}

Cardinal pay-off variant

In this version the recruiter's pay-off is given by the true value of the selected applicant.The interviewer's objective is to maximize the expected value of the selected applicant.

Just as in the classical problem, the optimal policy is given by a threshold, which for this problem we will denote by c, at which the interviewer should begin accepting candidates.

Bearden (2006) showed that c is:

$c = \lfloor \sqrt{n} \rfloor$ or $c = \lceil \sqrt{n} \rceil$ (In fact, whichever is closest)

Number of secretaries to choose greater than one

For another variant, the interviewer is not hiring just one secretary but rather is, say, admitting a class of students from an applicant pool.

Under the assumption that success is achieved if and only if all the selected candidates are superior to all of the not-selected candidates, n is even and the desire is to select exactly half the candidates, in Vanderbei 1980 it was shown that the optimal strategy yields a success probability of $\frac{1}{n/2+1}$

Experimental studies

Experimental psychologists and economists have studied the decision behavior of actual people in secretary problem situations.

In large part, this work has shown that people tend to stop searching too soon. This may be explained, at least in part, by the cost of evaluating candidates. In real world settings, this might suggest that people do not search enough whenever they are faced with problems where the decision alternatives are encountered sequentially.

How many people should we get to know before we choose our definitive partner?

Professor of the University of Sidney, Clio Creswell in her book "Mathematics and Sex", asks: How many people should we know, at least, before we choose our definitive partner?

The answer, as she says, is 12. That is, the best strategy is to choose as partner the next person that is "better" than those 12 candidates.

Basic Bibliography

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